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Algebraic equivalence between certain models for superfluid–insulator transition

Luigi Amico
Dipartimento di Metodologie Fisiche e Chimiche (DMFCI), Universitá di Catania, viale A. Doria 6, I-95125 Catania, Italy
Istituto Nazionale per la Fisica della Materia, Unitá di Catania, Italy

Algebraic contraction is proposed to realize mappings between models Hamiltonians. This transformation contracts the algebra of the degrees of freedom underlying the Hamiltonian. The rigorous mapping between the anisotropic XXZ Heisenberg model, the Quantum Phase Model, and the Bose Hubbard Model is established as the contractions of the algebra $u(2)$ underlying the dynamics of the XXZ Heisenberg model.

The problem of mapping between not equivalent algebras was solved, in mathematical physics, years ago by Inönü and Wigner [1] and subsequently generalized by Saletan [2] when they founded the concept of contraction of a Lie algebra $\mathcal{A}$ [3]. Algebraic contraction is a transformation which may be singular on $\mathcal{A}$'s basis (namely, the kernel of the transformation is non trivial), while it is regular on its commutation brackets [4]. Applications of algebraic contractions in condensed matter physics trace back to studies of Umezawa and coworkers [5]. They shown, under quite general hypothesis, that in a zero temperature phase transition, the symmetry of the system in the disordered phase (is *rearranged*) contracts (through contraction of the algebra spanned by degrees of freedom of the system), onto the symmetry of the ordered phase. For example, in Heisenberg ferromagnets the broken symmetry $so(3)$ (which is the spin algebra and which accounts for the rotation symmetry of the magnetization in the paramagnetic phase) is contracted onto the euclidean symmetry $e(2)$ of the traslators (which accounts for the traslational invariance and for the rotation symmetry around the magnetization axes of the ordered state) [5].

In the present work, I apply contractions in physics toward a slightly different direction. That is: Contractions of algebras spanned by the degrees of freedom of the system as establishing a link between models which are intrinsically distinct (in the sense they are not unitarly connectible). I will provide an application of such idea in condensed matter physics: I will show that contractions can provide exact mapping between the Bose Hubbard model, the quantum Josephson model and certain anisotropic Heisenberg model. The motivation is to found rigorously the relation between these three models, which is employed (using physical grounds) to describe low temperature behaviour in various mesoscopic systems characterized by superfluidity (for applications of theses three models in mesoscopic physics, see for istance Refs. [6,7]) The Bose Hubbard Model (BHM) describes a lattice gas of interacting charged bosons. Its elementary degrees of freedom are the bosonic site j annihilation a_j, creation $a_j^\dagger$, and number n_j operators. The Quantum Phase Model (QPM) is largely employed in the physics of Josephson junctions arrays since it can describe the competition between quantum phase coherence and Coulomb blockade. The elementary degrees of freedom entering the QPM are the phases of the superconducting order parameter ϕ_j and the charge unbalance to charge neutrality $N_j := -i\partial_{\phi_j}$ (its eigenvalues range in $(-\infty, +\infty)$) in the island j. These two variables are considered as canonically conjugated in the QPM.

The phase diagrams of BHM and QPM were analyzed by many authors [8]. They describe zero temperature quantum phase transitions between incompressible insulators and coherent superfluid phases.

Finally, the XXZ anisotropic Heisenberg model [9] shows a low temperature behaviour related to those ones of BHM and QPM. In particular, its zero temperature phase diagram shows phase transitions from paramagnetic to canted phases that can be interpreted as insulator to superfluid phase transition [10].

Up to the present study, the relation between the BHM, the QPM and the XXZ model consisted in the fact that they belong to the same universality class. Unitary transformations mapping one model on each other do not exists. In fact, the arguments usually employed to relate such models on each other did not want to be rigorous [8]. For instance, the phase–number variables entering the QPM cannot be thought as mathematically originated from bosonic operators in BHM since a *no–go* theorem forbids $a_j \sim \sqrt{n_j}e^{i\phi_j}$, $a_j^\dagger \sim e^{-i\phi_j}\sqrt{n_j}$ ("even with the widest reasonable latitude of interpretation" [11,12]) as long as the phases ϕ_j are hermitian and canonically conjugated to a bounded (from below) n_i (as it is the bosonic number operator). A way out from this difficulty is realized in QPM by removing the hypothesis of boundness from below of n_i. It is worthwhile noting that connections between n_i and N_i cannot be unitary since their spectra are not isomorphic. Even more, such two operators cannot be unitarly connected to spin operators since unitary transformations cannot transform bounded into unbounded operators. In contrast, algebraic contractions can do such a job. I will use it as the crucial tool to realize the mapping between the three models I deal with. Such a transformation induces also the mapping of the matrix elements of the Hamiltonians as well of the phase boundaries in their phase diagrams. Following this direction, contraction was employed in Ref. [13] to map the

zero temperature phase diagram of the BHM onto the phase diagrams of the QPM and of the XXZ model within (suitable) mean field approximation.

The paper is organized as follow. After having outlined the general procedure consisting in contracting the underlying algebra characterizing quite general Hamiltonians, then it is applied to realize the mapping between the BHM, the QPM, and the XXZ.

I assume models Hamiltonian on a lattice Λ writable in terms of generators of a given Lie algebra $\mathcal{A} = \oplus_{i\in\Lambda}\ g_i$ having the form:

$$H = \sum_{v,w}\sum_{i,j} h_{v,i}\xi_{i,j}h_{w,j} - \sum_{\alpha\neq\alpha'}\sum_{i,j}\left(e_{\alpha,i}\zeta_{ij}e_{\alpha',j} + e_{\alpha',j}\zeta_{ji}e_{\alpha,i}\right) , \tag{1}$$

where $\xi_{i,j}$ and ζ_{ij} are real parameters. The local algebra g_i is defined as $g_i := 1\!\!1 \otimes \ldots \otimes 1\!\!1 \otimes g \otimes 1\!\!1 \otimes \ldots \otimes 1\!\!1$ with g at the i–th lattice position; the sum on α's runs on the set of simple roots of g_i; $i, j \in \Lambda$. Any g is assumed a rank–r semisimple Lie algebra of dimension $dim(g)$ [14] whose generators, in the Cartan–Weyl normalization, obey the standard commutation rules: $[h_{v,i}, h_{w,i}] = 0$, $(v, w = 1 \ldots r)$, $[h_{v,i}, e_{\alpha,i}] = \alpha_v e_{\alpha,i}$, $[e_{\alpha,i}, e_{\beta,i}] = c^{\alpha+\beta}_{\alpha,\beta} e_{\alpha+\beta,i}$ if $\alpha + \beta \neq 0$ and $[e_{\alpha,i}e_{-\alpha,i}] = \alpha^v h_{v,i}$; $c^{\gamma}_{\alpha,\beta}$ are the structure constants (the sum convention is assumed). For the global algebra $\mathcal{A}$, contractions can be done as products of local contractions of each g_i [15]. That is, as transformation $R = R(\epsilon; p) := \prod_i R_i(\epsilon; p)$ where $R_i(\epsilon; p) := 1\!\!1 \otimes \ldots \otimes 1\!\!1 \otimes r(\epsilon; p) \otimes 1\!\!1 \otimes \ldots \otimes 1\!\!1$ (where ϵ is a real variable, and p is a real parameter). The matrix $R_i(\epsilon; p)$ maps g_i onto another algebra g'_i which is in one to one correspondence with g_i when $\epsilon \neq 0$; additionally, there exist the limit $\epsilon \to 0$ for any value of the parameter p:

$$\begin{aligned}
&\lim_{\epsilon\to 0}[h'_{v,i}, h'_{w,i}] = 0 , \\
&\lim_{\epsilon\to 0}[h'_{v,i}, e'_{\alpha,i}] = \alpha(\epsilon; p)_v e'_{\alpha,i} , \\
&\lim_{\epsilon\to 0}[e'_{\alpha,i}, e'_{\beta,i}] = c(\epsilon; p)^{\alpha+\beta}_{\alpha,\beta} e'_{\alpha+\beta,i} \quad \alpha + \beta \neq 0 \\
&\lim_{\epsilon\to 0}[e'_{\alpha,i}e'_{-\alpha,i}] = \alpha^v(\epsilon; p)h'_{v,i} .
\end{aligned} \tag{2}$$

The operators: $h'_{v,i} := r^v(\epsilon; p)h_{v,i}$ and $e'_{\alpha,i} := r^\alpha(\epsilon; p)e_{\alpha,i}$, where $r^v(\epsilon; p)$ and $r^\alpha(\epsilon; p)$ are submatrices of $r(\epsilon; p)$ acting only on Cartan subalgebra (spanned by the set of $h_{v,i}$, $v \in (1 \ldots r)$) and root space separately, (spanned by the set of $e_{\alpha,i}$, $\alpha \in (1 \ldots dim(g) - r)$) define the transformed basis of the new algebra $g'_i := R_i g_i$ $(dim(g'_i) \equiv dim(g_i))$. The algebra $\mathcal{A}' := R[\mathcal{A}]$ which may be not unitarily equivalent to $\mathcal{A}$, is the contraction of $\mathcal{A}$.
As result of the contraction $R[\mathcal{A}]$, the Hamiltonian (1) is transformed into the contracted Hamiltonian as

$$H \to H' := RH = \sum_{v,w}\sum_{i,j} h'_{v,i}\xi_{i,j}h'_{w,j} - \sum_{\alpha\neq\alpha'}\sum_{i,j}\left(e'_{\alpha,i}\zeta_{ij}e'_{\alpha',j} + h.c\right) , \tag{3}$$

where the set of new degrees of freedom $\{h'_{v,i}\, ,\, e'_{\alpha,i}\}$ does not have to be unitarily equivalent to the set of the original variables $\{h_{v,i}\, ,\, e_{\alpha,i}\}$.

Now I apply the scheme developed above to map the XXZ model on to BHM, and QPM [16]. In this case, it is sufficient to consider the algebra g_i having rank–1; thus the sum on simple roots in Hamiltonian (1) reduces to a single term coupling the positive with the negative root operators. The Hamiltonian (3) becomes:

$$H' = \rho\sum_i h'_i + \sum_{i,j} h'_i\xi_{ij}h'_j - \sum_{\langle i,j\rangle}\left(e'_{+,i}\zeta_{i,j}e'_{-,j} + e'_{+,j}\zeta_{j,i}e'_{-,i}\right) , \tag{4}$$

where the linear term in Cartan generators has been isolated; the sum of the roots operators $e'_{\pm\alpha,i} \equiv e'_{\pm,i}$ involves only nearest neighbouring site indices.
The algebra g'_i is to be taken as the rotated $R_i(\epsilon, p)[u(2)_i]$ of $u(2)_i = 1\!\!1 \oplus su(2)_i$ which is characterized by $[J^3_i, J^\pm_j] = \pm\delta_{i,j}J^\pm_i$, $[J^+_i, J^-_j] = 2\delta_{i,j}J^3_i$, $[\mathbf{J}_i, 1\!\!1] = 0$ with standard representations $J^3_i|J_i, m_i\rangle = m_i|J_i, m_i\rangle$

$J_i^{\pm}|J_i,m_i\rangle = [(J_i \mp m_i)(J_i \pm m_i+1)]^{1/2}\,|J_i,m_i\pm 1\rangle$. The matrix $R_i(\epsilon,p)$ defines the change of "basis" of $u(2)_i$ (as vector space) $(e'_{+,i}\ ,\ e'_{-,i}\ ,\ h'_i\ ,\ \mathbb{1})^T = r(\epsilon;p)(J_i^+\ ,\ J_i^-\ ,\ J_i^3\ ,\ \mathbb{1})^T$ with:

$$r(\epsilon;p) := \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 1 & \frac{p}{2\epsilon^2} \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{5}$$

The generators $\mathbf{h}'_i := \{h'_i\ ,\ e'_{\pm,i}\}$ are expressed in terms of $\mathbf{J}_i$ as

$$e'_{\pm,i} = \epsilon J_i^{\pm} \qquad\qquad h'_i = J_i^3 + \mathbb{1}\frac{p}{2\epsilon^2}\,. \tag{6}$$

The commutation rules of g_i are:

$$\begin{aligned} [h'_i, e'_{\pm,j}] &= \pm\delta_{i,j} e'_{\pm,i} \\ [e'_{+,i}, e'_{-,j}] &= \delta_{i,j}(2\epsilon^2 h'_i - p\mathbb{1}) \\ [\mathbf{h}'_i, \mathbb{1}] &= 0\ . \end{aligned} \tag{7}$$

The matrix elements of the Hamiltonian (4) are:

$$\begin{aligned} \langle J',m'|H'|J,m\rangle = \rho\sum_i B_i\delta_{m'_i,m_i} + &\sum_{i,j} B_i\xi_{ij}B_j\delta_{m'_i,m_i}\delta_{m'_j,m_j} \\ &- \sum_{\langle i,j\rangle}\Big(\zeta_{i,j}C_{i,j}\delta_{m'_i,m_i+1}\delta_{m'_j,m_j-1} + i \leftrightarrow j\Big)\ , \end{aligned} \tag{8}$$

where $|J,m\rangle := \otimes_i |J_i,m_i\rangle$, $B_i := m_i + \frac{p}{2\epsilon^2}$ and:
$C_{i,j} := \epsilon^2\sqrt{(m_i - J_i)(m_j + J_j)(m_i + J_i + 1)(m_j - J_j + 1)}$. A trivial case corresponds to leaving ϵ as finite and setting $p = 0$. In such a case, ϵ can be normalized; $r(\epsilon;p)$ is isomorphic to the identity: $(e'_{\pm,i}\ ,\ h'_i\ ,\ \mathbb{1}) \equiv (J_i^{\pm}\ ,\ J_i^3\ ,\ \mathbb{1})$. Thus, the resulting Hamiltonian (4) is the XXZ model where ρ, $\xi_{i,j}$, $\zeta_{i,j}$ can be interpreted as the external magnetic field and the magnetic exchange coupling constants respectively.
Instead, the contraction of $\oplus_i u(2)_i$ is realized through the limit $\epsilon \to 0$: The transformation R is singular, but the commutation rules (8) are well defined.
There are two possible choices: *i)*: $\epsilon \to 0$, $p = 0$; *ii)*: $\epsilon \to 0$, $p \neq 0$.
In the case *i)* the commutation rules (8) contract to:

$$\begin{aligned} [h'_i, e'_{\pm,j}] = \pm\delta_{i,j}e'_{\pm,i}\ ,\ [e'_{+,i}, e'_{-,j}] = 0 \\ [\mathbf{h}'_i, \mathbb{1}] = 0\ . \end{aligned} \tag{9}$$

Such commutation relations are isomorphic to the commutation relations of the algebra $e(2)_i \oplus \mathbb{R}$. Thus, the Hamiltonian (4) contracts to the QPM: $H_{QP} = \sum_{i,j}(N_i - N_x)\,V_{ij}\,(N_j - N_x) - E_J/2\sum_{<i,j>}\cos(\phi_i - \phi_j)$, where $N_x\sum_j V_{i,j} \equiv \rho\ \forall i$, $V_{i,j} \equiv \xi_{i,j}$, and $\delta_{<i,j>}E_J/2 \equiv \zeta_{i,j}$. Where $[N_i,\phi_j] = i\delta_{i,j}$. Such a commutation relation induces $[N_i, e^{\pm i\phi_j}] = \pm\delta_{i,j}e^{\pm i\phi_j}$; from the hermitianity of ϕ_j it comes also that: $[e^{+i\phi_i}, e^{-i\phi_j}] = 0$ (compare with (9)).
The representations of the contracted algebra [3,17] $e(2)_i$ are the contraction of the representations of $u(2)_i$ for large J_i: $\langle J_i,m_i|\epsilon J_i^{\pm}|J_i,m'_i\rangle \to l_i\delta_{m'_i,m_i\pm 1}$ requiring that $\epsilon J_i \to l_i$, $l_i = \epsilon J_i$ being finite real numbers; whereas $\langle J_i,m_i|J_i^3|J_i,m'_i\rangle \to N_i\delta_{m'_i,m_i}$ whose eigenvalues can range in $(-\infty,+\infty)$ after having done the limit $J_i \to \infty$. In fact, this contraction (of representations) can be seen as suitable large J ($J_i \equiv J\ ,\forall i$) limit of the Villain realization of spin algebra [13,18] $J_j^+ := e^{i\phi_j}\sqrt{(J+1/2)^2 - (J_j^3+1/2)^2}$, $J_j^- = (J_j^+)^\dagger$ where J_j^3 fulfills $[J_j^3, e^{\pm i\phi_l}] = \pm\delta_{j,l}e^{\pm i\phi_l}$.
In the Ref. [13] it is shown that J plays the role of the Cooper pairs density in the islands.
The matrix elements of the contracted Hamiltonian can be obtained through $B_i \equiv N_i$ and $C_{i,j} \to l_il_j\sqrt{1 - (N_iN_j/l_il_j)^2\epsilon^2}$ in (8).
In the case *ii)*, p can be normalized. The algebra resulting from the contraction of (8) is:

$$\begin{aligned} [h'_i, e'_{\pm,j}] = \pm\delta_{i,j}e'_{\pm,i}, [e'_{+,i}, e'_{-,j}] = \delta_{i,j}\mathbb{1} \\ [\mathbf{h}'_i, \mathbb{1}] = 0\,. \end{aligned} \tag{10}$$

Such commutations are isomorphic to the "single boson algebra" $(h_4)_i \oplus \mathbb{R}$: spanned by operators n_i, $a_i^\dagger$ and a_i fulfilling $[n_i, a_j] = -\delta_{i,j} a_i$, $[n_i, a_j^\dagger] = \delta_{i,j} a_i^\dagger$, $[a_i, a_j^\dagger] = \delta_{i,j}$ (compare with (10)). This set of operators are the microscopic operators of the BHM: $H_{BH} = -\mu \sum_i n_i + \sum_{i,j} n_i U_{i,j} n_j - \sum_{\langle i,j \rangle} (a_i^\dagger t_{j,i} a_j + a_j^\dagger t_{i,j} a_i)$, on which Hamiltonian (4) is contracted ($\mu \equiv -\rho$, $U_{i,j} \equiv \xi_{i,j}$, and $t_{i,j} \equiv \zeta_{i,j}$).
The representations of the contracted algebra (10) are $\langle J_i, m_i | \epsilon J_i^\pm | J_i, m_i' \rangle \to \sqrt{n_i + 1/2(1 \pm 1)} \delta_{n_i', n_i \pm 1}$ where $J_i + m_i \to n_i$ (keeped finite in the limit) and $2 J_i \epsilon^2 \to 1$ for $J_i \to \infty$, $m_i \to -\infty$; whereas $\langle J_i, m_i | J_i^3 + 1/(2\epsilon^2) \mathbb{1} | J_i, m_i' \rangle \to n_i \delta_{m_i', m_i}$. I point out that the matrix elements of the bosonic number operator are obtained renormalizing angular momentum's matrix elements by $1/\epsilon^2 \to \infty$ since m_i, originally ranging in $(-J_i \dots J_i)$, must cover the interval $(0 \dots \infty)$. In fact, $1/\epsilon^2 \sim J$; then, this contraction (of representations) can be seen as suitable large J limit of the spin algebra in the Holstein Primakoff realization [13,18]: $J_j^+ := \sqrt{2J} a_j^\dagger \sqrt{1 - n_j/(2J)}$, $J_j^- = (J_j^+)^\dagger$, $J_j^3 := n_j - J$. In the Ref. [13] it is shown that J can be interpreted as the bosons density.
The matrix elements of the contracted Hamiltonian can be obtained through $B_i \equiv n_i$ and $C_{i,j} \to \sqrt{(n_i + 1) n_j} \sqrt{1 - \epsilon^2}$ in (8).

The algebra $(h_4)_i$ can be contracted further. Such a contraction induces the mapping between the BHM and the QPM as follow.
The BHM Hamiltonian can be written trivially as Hamiltonian (4), whose algebra is the enveloping of g_i spanned by the transformed $R_i(\epsilon, 2p)[(h_4)_i]$, for $p = 0$, $\epsilon = 1$. For generic ϵ, g_i is spanned by the operators $(A_i^+ \,,\, A_i^- \,,\, A_i^3 \,,\, \mathbb{1})^T = r(\epsilon; p)(a_i^\dagger \,,\, a_i \,,\, n_i \,,\, \mathbb{1})^T$. The new generators $\mathbf{A}_i := \{A_i^\pm \,,\, A_i^3\}$ are expressed in terms of $\{a_i^\dagger \,,\, a_i \,,\, n_i\}$ as

$$A_i^+ = \epsilon a_i^\dagger \,,\; A_i^- = \epsilon a_i \,,\; A_i^3 = n_i + \frac{p}{\epsilon^2} \,, \tag{11}$$

whose commutation rules are

$$\begin{gathered} [A_i^3, A_j^\pm] = \pm \delta_{i,j} A_i^\pm \quad , \quad [A_i^+, A_j^-] = \epsilon^2 \mathbb{1} \\ [\mathbf{A}_i, \mathbb{1}] = 0 \,. \end{gathered} \tag{12}$$

The limit $\epsilon \to 0$ (with finite p) realizes the (local) contraction of $(h_4)_i \oplus \mathbb{R}$ in $e(2)_i \oplus \mathbb{R}$ and thus it induces the contraction of the underlying algebra of the BHM on the QPM's one.
I point out that since the generators $\mathbf{A_i}$ can be seen as contraction of the vectors $\mathbf{J_i}$, the QPM is recovered as "first order" contraction of the BHM but also as a "second order" contraction of the XXZ model. This implies, in particular, that the coupling constants of the BHM and the QPM are related as: $E_J \simeq \epsilon t$. This suggests how superfluidity should be enhanced in the BHM respect to the QPM.

In conclusion, the contractions of the algebra $\mathcal{A} = \oplus_i u(2)_i$ underlying XXZ model, realize the exact mapping between the BHM, QPM and XXZ model. Using representations of $\mathcal{A}$, this was already employed in the Ref. [13] to relate the zero temperature phase diagram of the XXZ model, with those ones of the BHM and QPM. In particular, identifyng the mapping between these three models was crucial to bypass the problem concerning the coherent state representation of the phase–number algebra which is the basic difficulty involved in the semiclassical representation of the QPM.
As noted by Umezawa [5], such a contraction limit corresponds to consider low energy physical regime of the spin problem. Thus, the BHM and the QPM can be considered as low energy effective descriptions of the XXZ model.
I point out that the algebras underlying the three models as well their spectra are left distinct by the transformation above since the latter is, in particular, not unitary. This feature should be considered positively since mappings based on contractions can connect distinct physical scenarios. As it was already noted in Ref. [19,20], for istance, the difference between the Casimir operators of $e(2)$ and of spin algebras motivates qualitative difference between QPM's and XXZ model's zero temperature phase diagrams: in the QPM one's a metallic phase can exist; in XXZ one's such a metallic phase cannot exist. In this sense, mappings based on contractions express relations which are "weaker" than those ones based on unitary mappings.
Mappings based on contractions can be applied toward two different directions. First, they might serve to group the set of all mutually contracted models in "equivalence classes" following the same procedure known in group theory. There, the classification of algebras was considerably simplified by contractions which reduce the number of eventually independent algebras [3]. In the same way, properties of models in the same equivalence class can be stated succinctly and perspicaciously, analoglously to what is done having introduced the concept of universality class in the theory of phase transition [21].
Finally, contractions might be applied to the theory of integrable systems: properties of integrable models might be related to properties corresponding to non integrable models. It is whorthwhile noting that the same procedure

could not be persecuted through unitary relations since the latter can connect properties of integrable models to corresponding properties of models which are still integrable. In particular, exact properties of one dimensional QPM and BHM (which resist to be exactly solved) might be argued from corresponding properties of the XXZ model which, instead, is integrable in one dimension. Work is in progress along this direction.

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